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Wave radiation and wave diffraction from a submerged body in a uniform current

By JOHN GRUE AND ENOK PALM
Department of Mechanics, University of Oslo, Blindern, Oslo 3, Norway

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Radiation and diffraction of free-surface waves due to a submerged body in a uniform current is considered. The fluid layer is infinitely deep and the motion is two-dimensional. Applying the method of integral equations, the radiation problem and the diffraction problem for a submerged circular cylinder are examined. For small speed U of the current a forced motion of a given frequency will give rise to four waves. It is shown, however, that, for a circular cylinder, an incoming harmonic wave gives rise to two waves only. Depending on the frequency, the new generated wave may be considered as a transmitted or a reflected wave. The mean second-order force is computed. For the radiation problem the first-order damping force is also obtained. It is shown that, for some values of the parameters, the damping force is negative. This result is closely related to the fact that a harmonic wave travelling upstream with a phase velocity less than U conveys negative energy downstream. The forces remain finite as $U\sigma/g$ ($\sigma \equiv$ the frequency, $g \equiv$ the acceleration due to gravity) approaches $\frac{1}{4}$.

1. Introduction

The intention of the present paper is to discuss the radiation and diffraction characteristics of a circular cylinder submerged in a uniform current under a free surface. Only the two-dimensional problem is considered, and all equations are linearized. The corresponding problem with no current present has been discussed in several papers. One of the most striking results in the diffraction problem is that a submerged body of circular contour gives no reflection (Dean 1948; Ursell 1950; Grue & Palm 1984). The radiation problem has been discussed by Ogilvie (1963). One of his results is that a circular cylinder describing a circular orbit in the clockwise direction, generates at large distances a harmonic wave to the right of the cylinder and no motion to the left. Correspondingly, if the circular orbit is described in the opposite rotational direction, a harmonic wave is generated to the left of the cylinder and no motion to the right. Reversing the wave motion, this means that a circular cylinder describing a circular orbit may absorb an incoming wave completely. This obviously may be of interest in connection with construction of devices for wave-energy extraction (Evans *et al.* 1979).

In most practical applications of wave reflection and wave radiation from a submerged body a current may also be present. This may, for example, be true if we consider the forces acting on the submerged bracings of an oil platform. The bracings are usually long compared with the diameter, and a two-dimensional theory is appropriate for incident waves with crests parallel to the cylinder axis. Another example is pipelines in the ocean.

Regarded from the frame of reference where the current is zero, the relative frame of reference, the problem we shall examine is wave reflection and wave radiation from a moving submerged body. Wave reflection and wave radiation from a floating body is a fundamental problem in the theory of ship motion. We believe that our results for the submerged body may also throw some light on the corresponding ship problem.

The occurrence of a current complicates the problem in various respects. The steady current, as well as the oscillating wave motion, may generate vortex shedding. These effects are disregarded here, where we shall focus our interest on the generation of waves. The current also gives rise to steady waves behind the body. These waves, which here will be denoted as lee waves, will be discussed shortly in connection with the restriction they put on the validity of the time-periodic solution.

A typical feature of waves on a uniform current is that a body oscillating with a given frequency generates several waves of different wavelengths. More precisely, for weak currents, i.e. for $\tau = U\sigma/g < \frac{1}{4}$ ($\sigma \equiv$ frequency, $g \equiv$ acceleration due to gravity, $U \equiv$ speed of the current), four waves are generated. For stronger currents, i.e. for $\tau > \frac{1}{4}$, two waves are generated. In the diffraction problem (with one incoming harmonic wave) the submerged body will normally generate the same type of waves as in the radiation problem. This, however, is not true for a submerged body of circular form (see below).

The radiation problem will be studied in §4, and the diffraction problem in §5. The solution will be obtained by use of integral equations. These will be solved partly by an approximate method, valid for deeply submerged bodies, and partly by a numerical method where no such assumptions are made. The approximate method gives the solution of the integral equation immediately. It is shown that the approximate method leads to a fair approximation for circular cylinders with centres placed at depths larger than about twice the radius.

In the radiation problem we shall study sway, heave and the case when the centre of the cylinder describes a circular orbit. The first-order damping force and the second-order steady force will be computed. It is found that the damping force may be negative for some values of the parameters. This result is shown to be due to the fact that one of the generated waves, the k_3 wave, conveys negative wave energy.

In the diffraction problem it will be shown that a submerged body of circular form behaves very specially, generating only *one* new wave for all values of τ. Thus for an incoming k_1 wave (k_2-wave) only a reflection wave with wavenumber k_2 (k_1) is generated. An incoming k_3 wave or k_4 wave will be split into a k_3 wave and a k_4 wave, with no reflection waves. From the definitions (3.4) and (3.5) of k_1, k_2, k_3, k_4, it follows that this result is a direct generalization of the classical result that a body of circular contour gives no reflection for $U = 0$.

2. The boundary-value problem

Let coordinates be taken with the origin O in the mean free surface of the fluid, with the x-axis horizontal and normal to the generators of the cylinder, and with the y-axis positive upwards. The fluid is assumed incompressible and the motion irrotational. The uniform velocity U of the water is horizontal and along the negative x-axis. The velocity $\boldsymbol{v}$ may then be written

$$\boldsymbol{v} = \nabla\Phi^* - U\boldsymbol{e}_x, \tag{2.1}$$

where $\boldsymbol{e}_x$ is the unit vector along the x-axis and Φ^* a velocity potential. Φ^* satisfies the two-dimensional Laplace equation

$$\nabla^2\Phi^* = 0. \tag{2.2}$$

The fluid layer will be assumed to be of infinite depth. The boundary condition at $y = -\infty$ is then

$$\lim_{y\to-\infty} \nabla\Phi^* = 0. \tag{2.3}$$

Furthermore, at the submerged body we must have

$$\frac{\partial\Phi^*}{\partial n} - Un_x = v_n, \tag{2.4}$$

where $\partial/\partial n$ denotes the normal derivative and n_x is the x-component of the normal vector $\boldsymbol{n}$ of the body, defined as positive into the fluid. v_n is the normal velocity of the body. In addition, the boundary condition at the free surface must be satisfied. We shall assume that this condition may be linearized. This is a valid approximation if either the body is deeply submerged or the body is slender. Since we here consider bodies of circular form, the first-mentioned condition is assumed satisfied. How deeply the body must be submerged for the assumption to be valid will be discussed later. The free-surface condition then takes the form

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)^2 \Phi^* + g\frac{\partial\Phi^*}{\partial y} = 0 \quad (y = 0), \tag{2.5}$$

where t denotes time. Besides the boundary conditions (2.3)–(2.5), the radiation conditions as $x \to \pm\infty$ must be fulfilled.

The solution of the problem is divided into one steady solution and one oscillating part proportional to $\exp(\mathrm{j}\sigma t)$ with j denoting the imaginary unit (see the first paragraph of §3). We therefore write

$$\Phi^* = -U\chi(x,y) + \Phi(x,y,t), \tag{2.6}$$

where

$$\Phi(x,y,t) = \mathrm{Re}_\mathrm{j}\, \phi(x,y)\exp(\mathrm{j}\sigma t). \tag{2.7}$$

(Re_j denotes the real part with respect to j.) We have

$$\nabla^2\chi = 0, \qquad \nabla^2\phi = 0. \tag{2.8), (2.9}$$

Assume that the cylinder is oscillating with its centre at

$$x = \mathrm{Re}_\mathrm{j}\,(\xi_x \exp(\mathrm{j}\sigma t)), \quad y + d = \mathrm{Re}_\mathrm{j}\,(\xi_y \exp(\mathrm{j}\sigma t)). \tag{2.10}$$

The boundary conditions applied at the mean position of the body surface S are, for χ,

$$\frac{\partial\chi}{\partial n} = -n_x, \tag{2.11}$$

and for ϕ (see e.g. Newman 1978, equation (3.28)),

$$\frac{\partial\phi}{\partial n} = \boldsymbol{n}\cdot(\mathrm{j}\sigma\boldsymbol{\xi} + U(\boldsymbol{\xi}\cdot\nabla)\nabla\chi), \tag{2.12}$$

where

$$\boldsymbol{\xi} = \xi_x \boldsymbol{e}_x + \xi_y \boldsymbol{e}_y.$$

($\boldsymbol{e}_y$ is the unit vector along the y-axis.) The last term in (2.12) accounts for the interaction between the steady and the oscillatory flow fields. The linearized boundary conditions at $y = 0$ are

$$U^2\frac{\partial^2\chi}{\partial x^2} + g\frac{\partial\chi}{\partial y} = 0 \quad (y = 0), \tag{2.13}$$

$$\left(\mathrm{j}\sigma - U\frac{\partial}{\partial x}\right)^2 \phi + g\frac{\partial\phi}{\partial y} = 0 \quad (y = 0). \tag{2.14}$$

The lee-wave problem has been discussed by Lamb (1932, p. 410), Havelock (1926, 1936) and Tuck (1965). We shall here mainly be interested in the periodic motion.

The mathematical problem will be solved by transforming it to an integral equation. This may be achieved by expressing ϕ as a source distribution over the boundary of the submerged body.

3. The integral equation

The velocity potential for a concentrated source of strength unity, the Green function, oscillating with a frequency σ and imbedded in a current of speed U, has been derived by Haskind (1954). It is appropriate to rewrite his expression for the velocity potential so it becomes the real part of a Green function which is an analytic function of $z = x+iy$ (we use two imaginary units, i in connection with the space variables and j with the time variable). The Green function for a concentrated source at $z = z_0$ is then found to be

$$G(z, z_0) = \frac{1}{2\pi}\left[\ln\frac{z-z_0}{z-\bar{z}_0} - (1-\mathrm{ij})\,F_1(z) - (1+\mathrm{ij})\,F_2(z)\right], \tag{3.1}$$

where†

$$F_1(z) = \frac{1}{(1-4\tau)^{\frac{1}{2}}}\left[\int_{\infty}^{z} \frac{\exp(-\mathrm{i}k_1(z-u))\,\mathrm{d}u}{u-\bar{z}_0} - \int_{-\infty}^{z} \frac{\exp(-\mathrm{i}k_2(z-u))\,\mathrm{d}u}{u-\bar{z}_0}\right], \tag{3.2}$$

$$F_2(z) = \frac{1}{(1+4\tau)^{\frac{1}{2}}}\left[\int_{\infty}^{z} \frac{\exp(-\mathrm{i}k_3(z-u))\,\mathrm{d}u}{u-\bar{z}_0} - \int_{\infty}^{z} \frac{\exp(-\mathrm{i}k_4(z-u))\,\mathrm{d}u}{u-\bar{z}_0}\right], \tag{3.3}$$

and a bar denotes complex conjungate. With

$$\nu = \frac{\sigma^2}{g}, \quad \tau = \frac{U\sigma}{g}, \tag{3.4}$$

The four wavenumbers are defined by

$$k_{1,2} = \frac{\nu}{2\tau^2}\,[1-2\tau \pm (1-4\tau)^{\frac{1}{2}}], \quad k_{3,4} = \frac{\nu}{2\tau^2}\,[1+2\tau \pm (1+4\tau)^{\frac{1}{2}}]. \tag{3.5}$$

For $\tau > \frac{1}{4}$, k_1 and k_2 become complex in i.

Before proceeding further, let us discuss the far-field solution of the concentrated source. Such a discussion is given elsewhere in the literature (see e.g. Peregrine 1976). We find it, however, appropriate to give a short survey of the solution here. It is seen from (3.1)–(3.5) for $\tau < \frac{1}{4}$ that the solution consists of four waves, viz one wave with wavenumber k_2 as $x \rightarrow \infty$ and three waves with wavenumbers k_1, k_3, k_4 at $x = -\infty$. For $\tau > \frac{1}{4}$ the solution consists of no waves at $x = \infty$ and two waves with wavenumbers k_3, k_4, at $x = -\infty$.

The various wavenumbers are found as solutions of (i) $\sigma = Uk \pm (gk)^{\frac{1}{2}}$ $(k = k_4, k_3)$ and (ii) $\sigma = (gk)^{\frac{1}{2}} - Uk$ $(k = k_1, k_2)$, where σ is positive and known. The four solutions are indicated in figure 1. It is seen from the figure that, in order to get four waves, σ must be less than a certain maximal value, i.e. $\tau < \frac{1}{4}$. Furthermore, in the relative frame of reference, both the k_1 wave and the k_2 wave have positive phase velocities that are larger than U. The k_1 wave has, however, a group velocity less than U, and

† The formulas (3.2) and (3.3) correspond to those given by Haskind. We find, however, that for $\tau > \frac{1}{2}$ the lower limits in the k_1 integral and the k_2 integral must be $\mathrm{i}\infty/k_1$ and $\mathrm{i}\infty/k_2$ respectively, to ensure that $F_1(z)$ is bounded in the entire fluid.

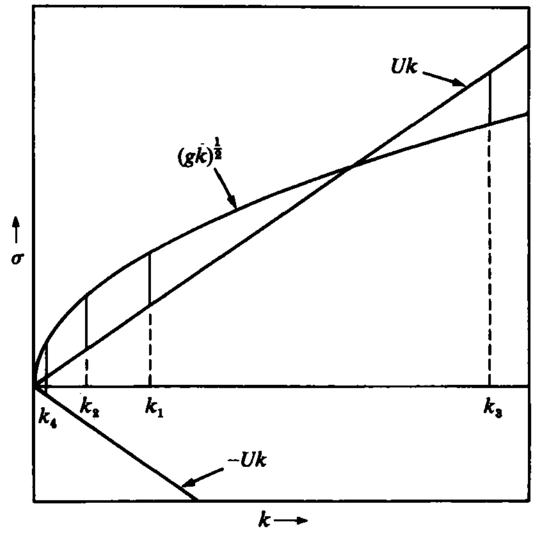

FIGURE 1. The four wavenumbers for given σ and U.

is therefore located downstream. The k_2 wave has a group velocity larger than U and is located upstream. The k_3 wave has positive phase velocity smaller than U, and the k_4 wave has negative phase velocity. These two waves are therefore located downstream. As $\tau \rightarrow \frac{1}{4}$ the k_1 wave and the k_2 wave merge into one wave with a group velocity equal to U. This leads to a singularity for $\tau = \frac{1}{4}$, revealed in the Green function (3.1), (3.2).

The velocity potential Φ for the motion in consideration may be written

$$\Phi(x, y, t) = \mathrm{Re}_i \, \mathrm{Re}_j \, f(z) \exp(j\sigma t), \tag{3.6}$$

where $f(z)$ is an analytic function of z ($f(z)$ is also complex in j). We write

$$f(z) = f_0(z) + f_1(z), \tag{3.7}$$

where in the diffraction problem $f_0(z)$ is the (known) complex potential of the incoming waves. In the radiation problem $f_0(z) = 0$. $f_1(z)$ must satisfy the radiation conditions at $x = \pm\infty$ and the boundary conditions at $y = 0$ and $-\infty$. For arbitrary, smooth contours we write $f_1(z)$ as a source distribution over S:

$$f_1(z) = \int_S \gamma(s) \, G(z, \zeta(s)) \, ds, \tag{3.8}$$

where s is the arclength and $z = \zeta(s)$ is the equation for the contour on parameter form. To secure that the boundary condition at $y = 0$ and the radiation condition are satisfied, γ must be real with respect to i. γ is, however, complex in j, being of the form $\gamma = \gamma_1 + j\gamma_2$, where γ_1 and γ_2 are real. γ has to be chosen so that the boundary conditions at the body is satisfied. Let $\beta(s)$ denote the angle between the tangent vector and the x-axis. Hereby

$$\exp(i\beta(s)) = \frac{d\zeta}{ds} \tag{3.9}$$

and

$$f'(\zeta(s)) \exp(i\beta(s)) = \frac{\partial\phi}{\partial s} + i\frac{\partial\phi}{\partial n}. \tag{3.10}$$

Letting $z \to \zeta$ and applying the boundary condition at the body, the Plemelj formula gives

$$\tfrac{1}{2}\mathrm{i}\gamma(s') + \fint_S \gamma(s)\, G'(\zeta(s'), \zeta(s)) \exp(\mathrm{i}\beta(s'))\, \mathrm{d}s = \frac{\partial \phi}{\partial s}(s') + \mathrm{i}\frac{\partial \phi}{\partial n}(s') - f_0'(\zeta(s')) \exp(\mathrm{i}\beta(s')). \quad (3.11)$$

A bar through the integral sign indicates the principal value. Taking the imaginary part with respect to i of this equation, we get rid of $\partial\phi/\partial s$, which is unknown. It is easily shown that this equation is non-singular. This follows from the fact that the only singular term may be written

$$\mathrm{Im}_\mathrm{i} \frac{\exp(\mathrm{i}\beta(s'))}{\zeta(s') - \zeta(s)} = \frac{\cos(n', r)}{r}, \quad (3.12)$$

where r is the radius vector from $\zeta(s)$ to $\zeta(s')$ and (n', r) is the angle between the radius vector and the outward normal at the point $\zeta(s')$ (see Grue & Palm 1984). Since $\cos(n', r) \to 0$ as $r \to 0$ the left-hand side of (3.12) is non-singular. The imaginary part of (3.11) becomes

$$\gamma(s') + \frac{1}{\pi}\int_S \gamma(s)\, L(s', s)\, \mathrm{d}s = h(s'). \quad (3.13)$$

Here

$$L = \mathrm{Im}_\mathrm{i}\Bigg[\exp(\mathrm{i}\beta(s'))\left(\frac{1}{\zeta(s') - \zeta(s)} - \frac{1}{\zeta(s') - \overline{\zeta(s)}}\right.$$
$$+ \frac{\mathrm{i}+\mathrm{j}}{(1-4\tau)^{\frac{1}{2}}}\left(k_1 \int_\infty^{\zeta(s')} \frac{\exp(-\mathrm{i}k_1(\zeta(s') - u))\, \mathrm{d}u}{u - \overline{\zeta(s)}} - k_2 \int_{-\infty}^{\zeta(s')} \frac{\exp(-\mathrm{i}k_2(\zeta(s') - u))\, \mathrm{d}u}{u - \overline{\zeta(s)}}\right)$$
$$\left.+ \frac{\mathrm{i}-\mathrm{j}}{(1+4\tau)^{\frac{1}{2}}}\left(k_3 \int_\infty^{\zeta(s')} \frac{\exp(-\mathrm{i}k_3(\zeta(s') - u))\, \mathrm{d}u}{u - \overline{\zeta(s)}} - k_4 \int_\infty^{\zeta(s')} \frac{\exp(-\mathrm{i}k_4(\zeta(s') - u))\, \mathrm{d}u}{u - \overline{\zeta(s)}}\right)\right)\Bigg] \quad (3.14)$$

and

$$h(s') = 2\frac{\partial \phi}{\partial n}(s') - 2\,\mathrm{Im}_\mathrm{i}\,(f_0'(\zeta(s')) \exp(\mathrm{i}\beta(s'))). \quad (3.15)$$

There is no net mass flux across the body surface, which sets restrictions on γ. To see this, we integrate (3.13) with respect to s' around the contour. Changing the order of integration, applying (3.9) and using Cauchy's theorem for analytic functions, we obtain

$$\int_S \gamma(s)\, \mathrm{d}s = \int_S \frac{\partial \phi}{\partial n}\, \mathrm{d}s = 0. \quad (3.16)$$

For a body of circular contour, $\cos(n', r)/r$ in (3.12) is constant. From (3.16) it then follows that the contribution from the first term in (3.14) vanishes.

Let R denote the radius of the cylinder and d the distance between the free surface and the centre of the cylinder. When $(d/R)^2 \gg 1$, the second term in (3.13) may be cancelled (see Grue & Palm 1984), and (3.13) reduces to

$$\gamma(s') = h(s'). \quad (3.17)$$

By using this simplification in the following sections, we are able to obtain convenient, analytical, solutions of the problem. We shall, however, also solve the complete equation (3.13), applying a numerical method. Generally speaking, it will turn out that our approximate solution is a fair approximation when d/R is larger than about 2, provided that τ is not close to $\frac{1}{4}$.

To solve the complete equation (3.13) we shall apply a Fourier transform. Introducing the angle variable θ instead of s, we have

$$\zeta(\theta) = \mathrm{i}R\exp(\mathrm{i}\theta) - \mathrm{i}d, \tag{3.18}$$

where $\theta = 0$ corresponds to the uppermost point of the circular contour. We also have

$$\beta(\theta) = \theta + \pi. \tag{3.19}$$

According to (3.16), $\gamma(\theta)$ may be written in the form

$$\gamma(\theta) = \sum_{m=1}^{\infty} (C_m \exp(\mathrm{i}m\theta) + \bar{C}_m \exp(-\mathrm{i}m\theta)). \tag{3.20}$$

As shown in Appendix A, the Fourier transform of (3.13) is appropriately written as two infinite sets of linear uncoupled equations. It turns out that the systems converge rather rapidly, and for all purposes in this paper it suffices to truncate the series after ten terms.

We shall especially be interested in the far field. By contour integration we obtain from (3.1)–(3.3) and (3.8) that

$$\lim_{x\to\infty} f_1(z) = A_2 \exp(-\mathrm{i}k_2 z), \tag{3.21}$$

$$\lim_{x\to-\infty} f_1(z) = A_1 \exp(-\mathrm{i}k_1 z) + A_3 \exp(-\mathrm{i}k_3 z) - A_4 \exp(-\mathrm{i}k_4 z), \tag{3.22}$$

where

$$A_q = \mathrm{i}(1-\mathrm{ij})\frac{1}{(1-4\tau)^{\frac{1}{2}}}\int_S \gamma(s)\exp(\mathrm{i}k_q\overline{\zeta(s)})\,\mathrm{d}s \quad (q = 1,2), \tag{3.23}$$

$$A_q = \mathrm{i}(1+\mathrm{ij})\frac{1}{(1+4\tau)^{\frac{1}{2}}}\int_S \gamma(s)\exp(\mathrm{i}k_q\overline{\zeta(s)})\,\mathrm{d}s \quad (q = 3,4). \tag{3.24}$$

Substituting the Fourier series (3.20) in (3.23) and (3.24), A_q is obtained in form of an infinite series (see Appendix A, (A 14) and (A 15)).

4. The oscillating cylinder

As a first application, we consider the motion generated by an oscillating submerged circular cylinder. We shall consider four cases: sway, heave and with the centre of the cylinder describing a circular orbit clockwise or counterclockwise.

The function $f_0(z)$ is now chosen as zero and the right-hand side of (3.13) reduces to $2\,\partial\phi/\partial n$, which is given by (2.12). Furthermore, we have

$$n_x = -\sin\theta, \quad n_y = \cos\theta. \tag{4.1}$$

Hence

$$2\frac{\partial\phi}{\partial n} = \xi_x\left(-2\mathrm{j}\sigma\sin\theta + 2U\frac{\partial^2\chi}{\partial n\,\partial x}\right) + \xi_y\left(2\mathrm{j}\sigma\cos\theta + 2U\frac{\partial^2\chi}{\partial n\,\partial y}\right). \tag{4.2}$$

It is easily shown, utilizing the boundary condition at the body, that

$$U\frac{\partial^2\chi}{\partial n\,\partial x} = -\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\theta}(v_t\sin\theta), \tag{4.3}$$

$$U\frac{\partial^2\chi}{\partial n\,\partial y} = \frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}\theta}(v_t\cos\theta), \tag{4.4}$$

where v_t is the total tangential velocity in the lee-wave problem, given by

$$v_t = -\frac{U}{R}\frac{\partial}{\partial\theta}(x+\chi). \tag{4.5}$$

Hence (4.2) only requires the knowledge of χ along the contour of the body.

By solving the χ-problem, the right-hand side of (3.13) or (3.17) is known. The far-field solution is found from (3.21)–(3.24). The elevation η of the free surface is found from the Bernoulli equation, assuming that the pressure is zero at the free surface. The free-surface elevation η_∞ at $x=\infty$ and $\eta_{-\infty}$ at $x=-\infty$ in the four cases of forced motion is found to be of the form (compare (3.1)–(3.5) and the following discussion)

$$\eta_\infty = a_2 \sin(k_2 x - \sigma t + \delta_2), \tag{4.6}$$

$$\eta_{-\infty} = a_1 \sin(k_1 x - \sigma t + \delta_1) + a_3 \sin(k_3 x + \sigma t + \delta_3) - a_4 \sin(k_4 x + \sigma t + \delta_4). \tag{4.7}$$

For $\tau > \frac{1}{4}$, a_1 and a_2 are zero.

We shall first find the amplitudes a_q and δ_q by applying the approximate method, where (3.13) is replaced by (3.17). This method has the merit that we obtain analytical expressions for the unknown quantities. In this approximation χ (near the contour) is easily found as

$$\chi = \mathrm{Re_i}\,\frac{R^2}{z+\mathrm{i}d}. \tag{4.8}$$

As expected, this is the velocity potential in the infinite-field solution, in which the cylinder is replaced by a dipole. From (4.5) it then follows that

$$v_t = 2U\cos\theta, \tag{4.9}$$

and the right-hand side of (3.17) is known. The values of a_q and δ_q (in this first approximation) are as follows.

Sway ($\xi_x = \epsilon,\ \xi_y = 0,\ \epsilon$ real):

$$a_q = \begin{cases} \dfrac{2\pi(k_q R)^2 \epsilon}{(1-4\tau)^{\frac{1}{2}}} \exp(-k_q d) \quad (\tau < \frac{1}{4},\ q = 1,2), & (4.10a) \\ 0 \quad (\tau > \frac{1}{4},\ q=1,2), & (4.10b) \\ \dfrac{2\pi(k_q R)^2 \epsilon}{(1+4\tau)^{\frac{1}{2}}} \exp(-k_q d) \quad (q = 3,4), & (4.10c) \end{cases}$$

$$\delta_q = \quad -\tfrac{1}{2}\pi \quad (q = 1,2,3,4). \tag{4.10d}$$

Heave ($\xi_x = 0,\ \xi_y = \epsilon$):

$$a_q \quad \text{given by } (4.10a\text{–}c),$$

$$\delta_q = 0 \quad (q = 1,2,3,4). \tag{4.11}$$

With the centre describing a circular orbit clockwise ($\xi_x = \epsilon, \xi_y = \mathrm{j}\epsilon$):

$$a_q = \begin{cases} \dfrac{4\pi(k_q R)^2 \epsilon}{(1-4\tau)^{\frac{1}{2}}} \exp(-k_q d) \quad (\tau < \frac{1}{4},\ q = 1,2), & (4.12a) \\ 0 \quad (\tau > \frac{1}{4},\ q=1,2), & (4.12b) \\ 0 \quad (q = 3,4), & (4.12c) \end{cases}$$

δ_q given by (4.10d).

With the centre describing a circular orbit counterclockwise ($\xi_x = \epsilon, \xi_y = -\mathrm{j}\epsilon$):

$$a_q = \begin{cases} 0 \quad (q = 1, 2), & (4.13a) \\ \dfrac{4\pi(k_q R)^2 \epsilon}{(1+4\tau)^{\frac{1}{2}}} \exp(-k_q d) \quad (q = 3, 4), & (4.13b) \end{cases}$$

δ_q given by (4.10d).

A characteristic feature for the approximate far-field solution for sway and heave, is that the phase difference is exactly $\frac{1}{2}\pi$, just as for $U = 0$. The formulas (4.12) and (4.13) may be compared with the corresponding ones for $U = 0$, obtained by Ogilvie (1963). We notice that with a current the result is rather different for the two directions of revolution. Thus for clockwise revolution we find for $\tau < \frac{1}{4}$ one wave upstream and one wave downstream. For $\tau > \frac{1}{4}$ we find no waves at all. For counterclockwise revolution, however, we obtain no waves at $x = \infty$ and two waves at $x = -\infty$.

We see from (4.10) that all four amplitudes depend on k in the same way, being all proportional to $k^2 \exp(-kd)$. Hence all the amplitudes have a maximum value for $kd = 2$. Therefore a_3, for example, is dominating when σ and U have values rendering $k_3 d$ close to 2, with corresponding results for the three other waves.

To solve the complete equation (3.13) we apply Fourier transforms, as sketched in Appendix A, and find the solution by standard numerical technique. The numerical solution shows that the approximate solution discussed above is a fair approximation when $d/R > 2$, say, and τ is not close to $\frac{1}{4}$. Thus we find in the numerical solution that in sway and heave the amplitudes are approximately equal and that the phase difference is close to $\frac{1}{2}\pi$. The amplitudes for the sway problem, obtained by both methods, are compared in figures 2(a, b).

For $\tau \to \frac{1}{4}$ the amplitudes a_1 and a_2 of the approximate solution tend towards infinity, as seen from (4.10a). The numerical solution shows, however, that a_1 and a_2 approach the same *finite* limit for $\tau \to \frac{1}{4}$, as indicated in figures 2(a, b). This is an unexpected result. The Green function (3.1)–(3.3) becomes infinite for $\tau \to \frac{1}{4}$, and this is apparently also true for A_1, A_2 (and thereby a_1, a_2) from (3.23). However, we see from the series expansion (A 14) that A_1 and A_2 are only dependent on $C_m^{(1)} - C_m^{(4)}$ and $C_m^{(2)} + C_m^{(3)}$, which are obtained from the infinite set of equations (A 1). Furthermore, it follows from (A 3) that $C_m^{(1)} - C_m^{(4)}$ and $C_m^{(2)} + C_m^{(3)} \to O((1-4\tau)^{\frac{1}{2}})$ for $\tau \to \frac{1}{4}$. Hence A_1 and A_2 tend towards a finite, common limit for $\tau \to \frac{1}{4}$. It may also be seen that A_3 and A_4 are smooth functions at $\tau = \frac{1}{4}$.

Thus we have obtained that a wave generated by an oscillating submerged cylinder, and moving upstream with a group velocity approaching the uniform current, has a finite amplitude. For the very special value for τ, $\tau = \frac{1}{4}$, however, the problem is undetermined.

We shall then examine the first-order damping force and the steady horizontal second-order force for sway and heave. The damping force is found from the far-field solution by applying the energy theorem. The velocity of the body in sway or heave is $-\sigma\epsilon \sin \sigma t$. The damping force may be written

$$F = D \sin \sigma t, \tag{4.14}$$

where D is the transfer function, different for sway and heave. The mean work performed by the damping force on the fluid is then

$$W = \tfrac{1}{2}\epsilon\sigma D. \tag{4.15}$$

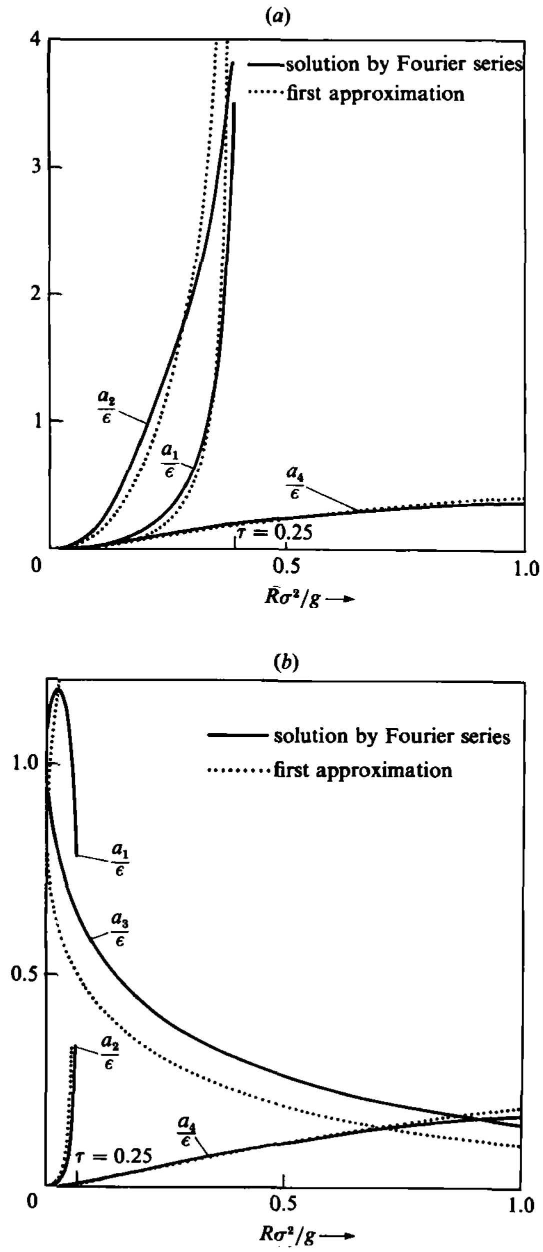

FIGURE 2. (*a*) Amplitudes for radiated waves in the sway problem, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 0.4$. a_3 is approximately zero. (*b*) Amplitudes for radiated waves in the sway problem, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 1.0$.

According to the energy equation averaged over a period, we have

$$W = R_{\infty} - R_{-\infty}, \tag{4.16}$$

where R_x denotes the energy flux. The energy flux for a single harmonic wave on a uniform current has been derived by Longuet-Higgins & Stewart (1960). We find it appropriate, however, to derive another version of the formula for the energy flux, dividing this in two parts. It is shown in Appendix B that for a single harmonic wave the energy flux R_x may (also for finite depth) be written

$$R_x = \left(\frac{\hat{p}}{\rho}+\tfrac{1}{2}U^2\right)M + c'_g E', \quad c'_g = c_g - U, \tag{4.17}$$

where M is the mean mass flux, ρ the density of the fluid, and $\hat{p}$ the constant second-order part of the pressure (being zero for infinite depth). Furthermore, here

$$E' = E - U\overline{u^{(1)}\eta^{(1)}} = E\frac{c-U}{c}, \tag{4.18}$$

where $u^{(1)} = \partial\Phi/\partial x$ at $y = 0$, $\eta^{(1)}$ is the first-order elevation and a bar denotes the time average. E is the wave-energy density in the relative frame of reference, given by $\frac{1}{2}\rho g a^2$, where a is the amplitude of the wave. c and c_g are respectively phase and group velocities, in the relative frame of reference. E' is the energy density transported with the group velocity c'_g and is the sum of E and the coupling energy of the uniform current and the wave. In other problems in fluid dynamics (and plasma physics) the quantity corresponding to E' is often called the wave-energy density, and $c'_g E'$ is the flux of wave energy. For a discussion of this concept, see, for example, Acheson (1974) and Cairns (1979). Here we only note that E' is not positive-definite.

Using the fact that the mean mass flux is independent of x, we obtain

$$W = \begin{cases} E'_2(c_{g2}-U) - E'_1(c_{g1}-U) - E'_3(c_{g3}-U) - E'_4(c_{g4}-U) & (\tau < \frac{1}{4}), \quad (4.19a) \\ -E'_3(c_{g3}-U) - E'_4(c_{g4}-U) & (\tau > \frac{1}{4}), \quad (4.19b) \end{cases}$$

where

$$E'_q = \tfrac{1}{2}\rho g a_q^2 \frac{c_q - U}{c_q} \quad (q = 1,2,3,4), \tag{4.20a}$$

$$c_q = \left(\frac{g}{k_q}\right)^{\frac{1}{2}} \quad (q = 1,2,3), \quad c_4 = -\left(\frac{g}{k_4}\right)^{\frac{1}{2}}, \quad c_{gq} = \tfrac{1}{2}c_q \quad (q = 1,2,3,4), \tag{4.20b}$$

Mathematically, (4.19) may be further reduced. Applying (3.4), (3.5) and (4.15), D may be written

$$D = \frac{\rho g}{2\epsilon}\left[\left(\frac{a_1^2}{k_1}+\frac{a_2^2}{k_2}\right)(1-4\tau)^{\frac{1}{2}} + \left(-\frac{a_3^2}{k_3}+\frac{a_4^2}{k_4}\right)(1+4\tau)^{\frac{1}{2}}\right]. \tag{4.21}$$

When $\tau > \frac{1}{4}$ the contribution from the k_1 wave and the k_2 wave cancel.

Values of D based on (4.21) are displayed in figures 3(a,b) for $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 0.4$ and $U/(gR)^{\frac{1}{2}} = 1.0$ respectively. We note that the approximate solution gives a fair approximation if τ is not close to $\frac{1}{4}$.

We also see that, in opposition to the approximate solution, the solution of the complete integral equation (3.13) gives values of D for $\tau \to \frac{1}{4}$ that are finite and independent of whether the limit is reached from below or above. This result follows immediately from (4.21), since a_1 and a_2 are finite. For larger values of d/R the damping transfer function may, however, vary rather rapidly close to $\tau = \frac{1}{4}$, as shown in figure 3(c).

Figure 3(b) reveals that, in a certain σ-domain, there is a negative damping. It is of interest to interpret this phenomenon in light of the formulas derived above. From (4.21) we note that only the k_3 wave can give rise to negative damping. Physically speaking, this means that the k_3 wave has a wave-energy flux *towards* the body, whereas the other three waves have wave-energy fluxes *away* from the body. When negative damping occurs the k_3 wave must predominate. Since the wave-energy

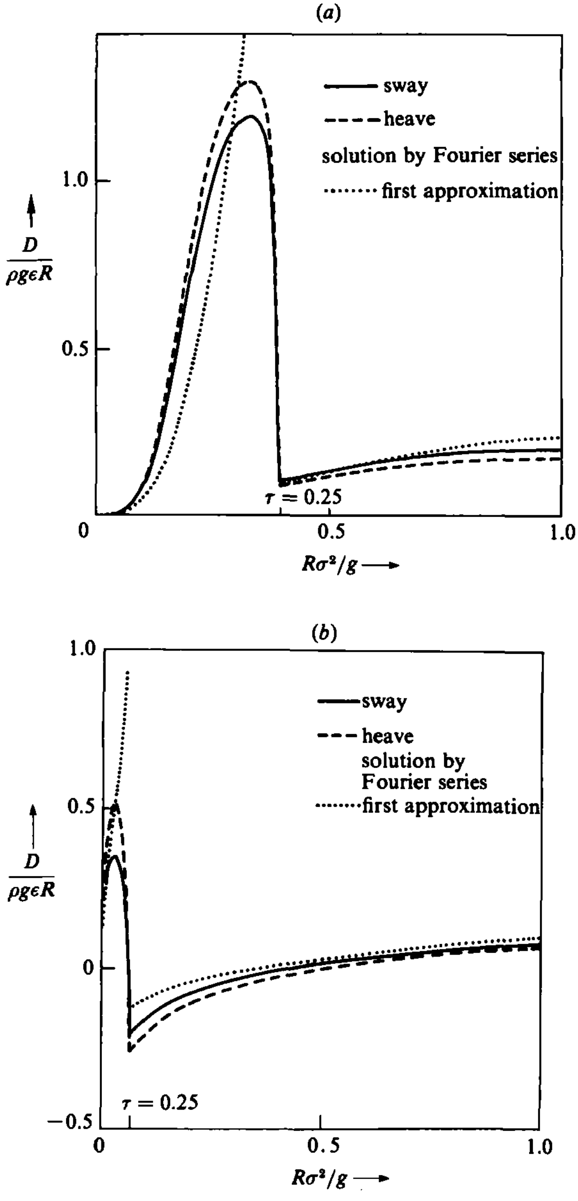

FIGURE 3(*a*–*b*). For caption see facing page.

flux for the k_3 wave is directed towards the body, i.e. in the positive x-direction (this wave is located downstream), and the group velocity $c_{g3} - U$ is along the negative x-direction, it follows that the wave-energy density E_3' is negative. This is confirmed by (4.18), since $c_3 < U$.

To derive the mean horizontal second-order force $\overline{F}_x$, we apply the momentum equation. We have

$$-\overline{F}_x = \overline{I}_\infty - \overline{I}_{-\infty}, \tag{4.22}$$

where I_x is the momentum flux defined by

$$I_x = \int_{-\infty}^{\eta} (\rho(u^{(1)} + u^{(2)} - U)^2 + p)\, \mathrm{d}y \tag{4.23}$$

(c)

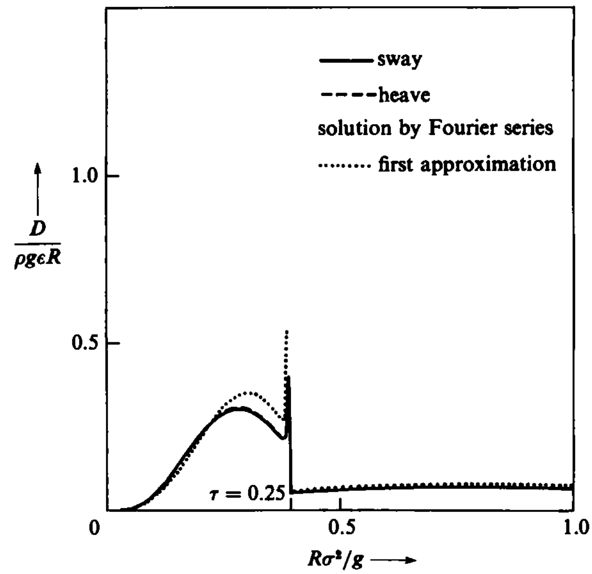

FIGURE 3. (*a*) Damping transfer functions in sway and heave, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 0.4$. (*b*) Damping transfer functions in sway and heave, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 1.0$. (*c*) Damping transfer functions in sway and heave, $d/R = 3.0$, $U/(gR)^{\frac{1}{2}} = 0.4$.

and $u^{(2)}$ is a second-order velocity whose time average does not vanish. Using the fact that the mass flux is independent of x, we obtain that

$$\bar{F}_x = -\frac{E_2}{c_2}(c_{g2}-U)+\frac{E_1}{c_1}(c_{g1}-U)+\frac{E_3}{c_3}(c_{g3}-U)+\frac{E_4}{c_4}(c_{g4}-U), \tag{4.24}$$

where
$$E_q = \tfrac{1}{2}\rho g a_q^2 \quad (q = 1,2,3,4). \tag{4.25}$$

When $\tau > \frac{1}{4}$ the contributions from the k_1 wave and k_2 wave cancel. If the lee waves are taken into account, the term

$$-\tfrac{1}{4}\rho g a_0^2 \tag{4.26}$$

has to be added to (4.24), where a_0 is the amplitude of the lee waves. The mean second-order force $\bar{F}_x$ is displayed in figure 4 for $d/R = 2.0$ and $U/(gR)^{\frac{1}{2}} = 0.4$. The force is small except for τ-values close to $\frac{1}{4}$. It is seen from the figure and it follows from (4.24) that $\bar{F}_x$ based on the complete solution approaches a finite value for $\tau \rightarrow \frac{1}{4}$, independent of whether the limit is obtained from below or above.

5. The diffraction problem

As the next application we consider the diffraction of an incoming harmonic wave due to a restrained submerged cylinder. Let the incoming wave elevation η_0 be given by

$$\eta_0 = a \sin(kx \pm \sigma t). \tag{5.1}$$

The corresponding velocity potential is

$$\Phi_0 = a\left(\frac{g}{k}\right)^{\frac{1}{2}} \delta \exp(ky)\cos(kx \pm \sigma t) = \mathrm{Re}_i\,\mathrm{Re}_j\, f_0(z)\exp(\mathrm{j}\sigma t). \tag{5.2}$$

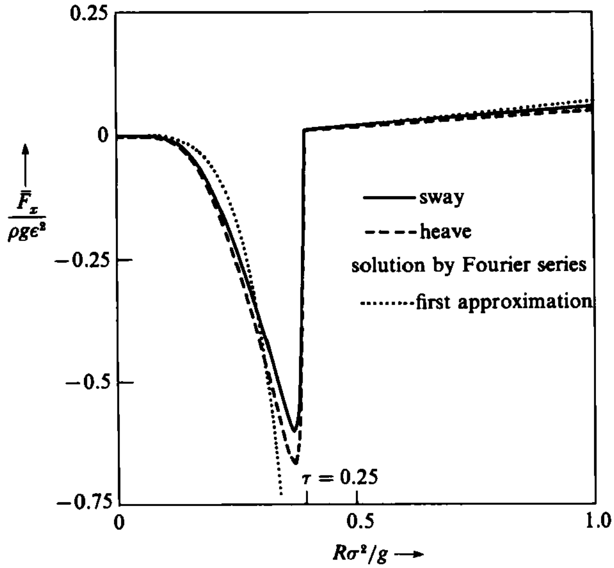

FIGURE 4. Mean second-order horizontal force in sway and heave, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 0.4$.

Here $\delta = 1$ for $k = k_4$, and $\delta = -1$ for $k = k_1, k_2, k_3$. $f_0(z)$ is given by

$$f_0(z) = \delta a\left(\frac{g}{k}\right)^{\frac{1}{2}}(1 \pm \mathrm{ij})\exp(-\mathrm{i}kz). \tag{5.3}$$

The $\pm$ sign indicates that the phase velocity may be along the negative or positive x-axes respectively.

For a submerged body of arbitrary form there will be a transmitted wave of the same wavenumber as the incoming wave. In addition, we expect that three new waves are generated for $\tau < \frac{1}{4}$ and one new wave for $\tau > \frac{1}{4}$. Let us first find the mean horizontal second-order force. A formula for this for a submerged body of arbitrary form may be written down immediately from (4.24). For example, let the incoming wave be a k_4 wave. The mean second-order force is then obtained from (4.24) by adding a term

$$-\frac{E_a}{c_4}(c_{g4} - U), \tag{5.4}$$

where $E_a = \frac{1}{2}\rho g a^2$. Hence

$$\bar{F}_x = -\frac{E_2}{c_2}(c_{g2} - U) + \frac{E_1}{c_1}(c_{g1} - U) + \frac{E_3}{c_3}(c_{g3} - U) + \frac{E_4 - E_a}{c_4}(c_{g4} - U). \tag{5.5}$$

For $\tau > \frac{1}{4}$ the contributions from the k_1 wave and the k_2 wave cancel. Correspondingly, if the incoming wave is a k_1 wave, k_2 wave or k_3 wave, $\bar{F}_x$ is obtained from (4.24) by adding

$$-\frac{E_a}{c_1}(c_{g1} - U), \quad \frac{E_a}{c_2}(c_{g2} - U), \quad -\frac{E_a}{c_3}(c_{g3} - U) \tag{5.6}$$

respectively. For later reference we also write down the energy equation, which is obtained from (4.16) by setting $W = 0$ and adding the contribution from the incoming wave. When this is a k_4 wave, we must add the term

$$E'_a(c_{g4} - U), \quad E'_a = E_a\frac{c_4 - U}{c_4}, \tag{5.7}$$

and we obtain

$$E_2'(c_{g2}-U)-E_1'(c_{g1}-U)-E_3'(c_{g3}-U)-(E_4'-E_a')(c_{g4}-U) = 0. \tag{5.8}$$

For $\tau > \frac{1}{4}$ the contributions from the k_1 wave and k_2 wave cancel. Similar expressions may be derived for incoming k_1 wave, k_2 wave and k_3 wave by adding the energy flux of the proper incoming wave.

In the formulas above we have considered a submerged body of arbitrary form. Let us now utilize the fact that the submerged body is a circular cylinder. We consider first the case where the incoming wave is a k_3 wave or a k_4 wave. We then have to use the plus sign in (5.2). The Fourier transform of $f_0(z)$ is given in Appendix A by (A 12), which according to (A 6) gives $h_n^{(1)}-h_n^{(4)} = h_n^{(2)}+h_n^{(3)} = 0$. Hence, from (A 1), $C_n^{(1)}-C_n^{(4)} = C_n^{(2)}+C_n^{(3)} = 0$. Introducing this result in (A 14), we obtain that $A_1 = A_2 = 0$. We have thus shown that the far-field solution consists of two waves only, viz a k_3 wave and a k_4 wave. For example, let the incoming wave be a k_4 wave. The solution is then of the form

$$\eta_\infty = a \sin(k_4 x+\sigma t), \tag{5.9}$$

$$\eta_{-\infty} = a_3 \sin(k_3 x+\sigma t+\delta_3)+a_4 \sin(k_4 x+\sigma t-\delta_4), \tag{5.10}$$

where δ_3 and δ_4 are phase constants. The solution for an incoming k_3 wave is obtained from (5.9) and (5.10) by replacing k_4 with k_3 in (5.9). We notice that the incoming wave is split up into two waves, but there is no reflection. The result is valid for all τ-values.

We then assume that the incoming wave is either a k_1 wave or a k_2 wave. We must then use the minus sign in (5.2). It is now found (from Appendix A) that $h_n^{(1)}+h_n^{(4)} = h_n^{(2)}-h_n^{(3)} = 0$. From (A 2) it then follows that $C_n^{(1)}+C_n^{(4)} = C_n^{(2)}-C_n^{(3)} = 0$. Introducing this in (A 15), we conclude that $A_3 = A_4 = 0$. In this case also the far field consists of two waves only. If the incoming wave is a k_1 wave at $x = \infty$ the solution will be of the form

$$\eta_\infty = a \sin(k_1 x-\sigma t)+a_2 \sin(k_2 x-\sigma t+\delta_2), \tag{5.11}$$

$$\eta_{-\infty} = a_1 \sin(k_1 x-\sigma t+\delta_1). \tag{5.12}$$

For an incoming k_2 wave at $x = -\infty$ we have

$$\eta_\infty = a_2 \sin(k_2 x-\sigma t+\delta_2), \tag{5.13}$$

$$\eta_{-\infty} = a_1 \sin(k_1 x-\sigma t+\delta_1)+a \sin(k_2 x-\sigma t). \tag{5.14}$$

It is here assumed that $\tau < \frac{1}{4}$. Hence for incoming k_1 waves and k_2 waves the motion consists of an incoming wave, a transmitted wave and a reflected wave. We have thus obtained, without making any approximation, that, for submerged bodies of circular contour, only two waves will occur in the diffraction problem. This result is a generalization to $U \neq 0$ of the fact that for $U = 0$ there is no reflection from a submerged circular cylinder (see the introduction). This latter result is easily obtained from the formulas above by letting $U \to 0$.

Let us now compute the amplitude of the reflected and transmitted wave. First we apply the approximate method based on (3.17). The Fourier transform of γ is then equal to (A 12). This introduced in (A 14) and (A 15) leads to the following expression for the amplitudes of the new waves:

$$a_q = \begin{cases} a \dfrac{4\pi k_q R}{(1-4\tau)^{\frac{1}{2}}} \exp(-(k_1+k_2)d)\, I_1(2(k_1 k_2)^{\frac{1}{2}} R) \quad (q = 1,2), & (5.15) \\ a \dfrac{4\pi k_q R}{(1+4\tau)^{\frac{1}{2}}} \exp(-(k_3+k_4)d)\, I_1(2(k_3 k_4)^{\frac{1}{2}} R) \quad (q = 3,4), & (5.16) \end{cases}$$

where I_1 denotes the modified Bessel function of the first kind of order one. Furthermore, the phase constants of the new k_1, k_2 and k_3 waves are $\frac{1}{2}\pi$, whereas the phase constant of the new k_4 wave is $-\frac{1}{2}\pi$. The phase constants of the transmitted waves are

$$\delta_q = \begin{cases} \dfrac{4\pi k_q R}{(1-4\tau)^{\frac{1}{2}}} \exp(-2k_q d)\, I_1(2k_q R) \quad (q = 1,2), & (5.17) \\ \dfrac{4\pi k_q R}{(1+4\tau)^{\frac{1}{2}}} \exp(-2k_q d)\, I_1(2k_q R) \quad (q = 3,4). & (5.18) \end{cases}$$

The amplitude of the transmitted wave in this approximation is found to be equal to the amplitude of the incoming wave.

We then compute the amplitudes by solving the complete equation (3.13) using the method outlined in Appendix A. The relative amplitudes of the generated k_3 wave for an incoming k_4 wave, based on (3.13) and (3.17), are displayed in figure 5(*a*). It is noted that for the considered values of the parameters the approximate solution is a fair approximation. The amplitude a_4 of the transmitted wave is found to be very close to a. The amplitudes of the reflected and transmitted waves for an incoming k_2 wave based on the complete equation (3.13) are displayed in figure 5(*b*). Since the energy flux for the incoming k_2 wave tends to zero as $\tau \rightarrow \frac{1}{4}$, we expect that the amplitudes a_1 and a_2 remain finite as $\tau \rightarrow \frac{1}{4}$. This is confirmed by the numerical computations.

The general expression (5.5) for the second-order force and the energy equation (5.8) takes a simpler form for a body of circular contour. Considering an incoming k_4 wave, the energy equation reduces to

$$a_4^2 - a^2 = a_3^2 \frac{k_4}{k_3}. \qquad (5.19)$$

We note that the amplitude of the transmitted wave is larger than the amplitude of the incoming wave, in spite of the fact that a new wave is generated. Using (5.19), the formula for the second-order force reduces to

$$\bar{F}_x = -\tfrac{1}{4}\rho g (1+4\tau)^{\frac{1}{2}} \left(1 - \frac{k_4}{k_3}\right) a_3^2. \qquad (5.20)$$

For an incoming k_3 wave the energy equation is obtained from (5.19) by interchanging the subscripts 3 and 4. It is found that

$$\bar{F}_x = -\tfrac{1}{4}\rho g (1+4\tau)^{\frac{1}{2}} \left(\frac{k_3}{k_4} - 1\right) a_4^2. \qquad (5.21)$$

For an incoming k_1 wave the energy equation gives

$$a_1^2 - a^2 = -a_2^2 \frac{k_1}{k_2}. \qquad (5.22)$$

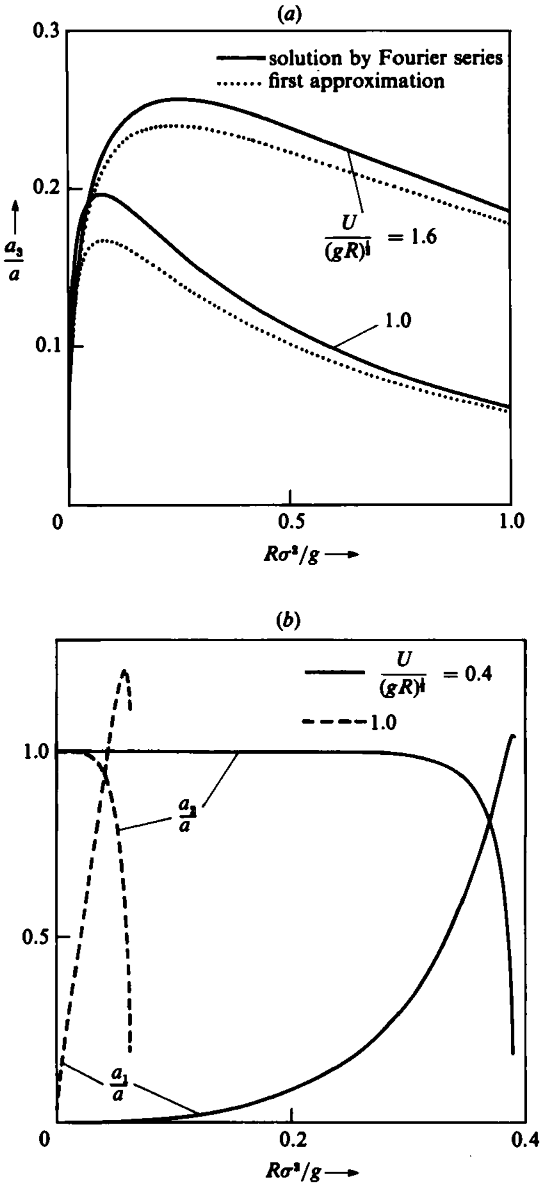

FIGURE 5. (*a*) Amplitude of new k_3 wave in the diffraction problem when k_4 waves are incident upon the cylinder, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 1.0, 1.6$. (*b*) Amplitude of transmitted k_2 wave and reflected k_1 wave when k_2 waves are incident upon the cylinder, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 0.4, 1.0$.

The second-order force may be written

$$\overline{F}_x = \tfrac{1}{4}\rho g(1-4\tau)^{\frac{1}{2}}\left(\frac{k_1}{k_2}-1\right)a_2^2. \tag{5.23}$$

It is noted that the force is positive, in spite of an incoming wave travelling downstream. If the incoming wave is a k_2 wave the energy equation becomes

$$a_2^2 - a^2 = -a_1^2\,\frac{k_2}{k_1} \tag{5.24}$$

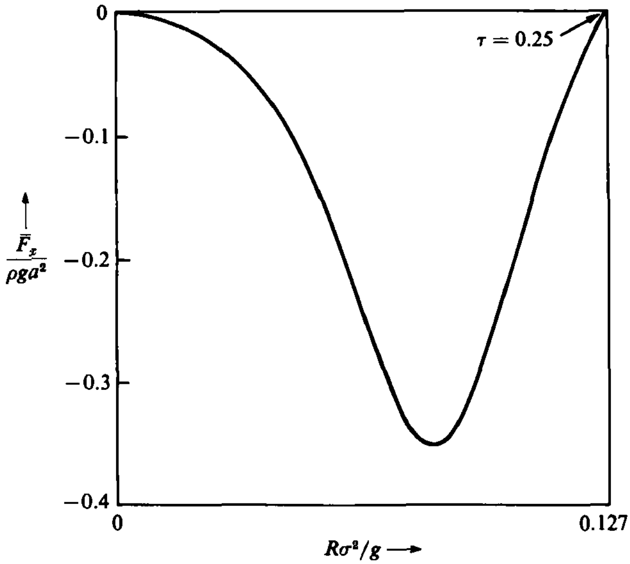

FIGURE 6. Mean second-order horizontal force when k_2 waves are incident upon the cylinder, $d/R = 2.0$, $U/(gR)^{\frac{1}{2}} = 0.7$.

and
$$\bar{F}_x = -\tfrac{1}{4}\rho g(1-4\tau)^{\frac{1}{2}}\left(1-\frac{k_2}{k_1}\right)a_1^2. \tag{5.25}$$

The force is now negative, whereas the incoming wave is travelling along the positive x-axis.

It is of interest to compare the second-order force $\bar{F}_x$ with the lee-wave force (4.26). For most values of the parameters, $\bar{F}_x$ is relatively small. It is found, however, that the largest force is obtained for an incoming k_2 wave. This force has a maximum value of about $0.35\rho g a^2$ for $R\sigma^2/g = 0.08$ and $U/(gR)^{\frac{1}{2}} = 0.7$ (see figure 6). We notice that, if the amplitude of the incoming wave is about the same as the amplitude of the lee wave, the maximal $\bar{F}_x$ is of the same magnitude as the lee-wave force.

6. Discussion

To examine more closely the validity of our solution, we have computed the lee-wave field and found the maximum value of $|\partial\chi/\partial x|$ at the free surface. It turns out that the maximum value is close to the maximum value at $x = -\infty$, which is displayed in figure 7. In order that the solution be a uniformly good approximation, this maximum value should be considerably smaller than 1. We notice from the figure that the maximum of $|\partial\chi/\partial x|$ is in fact larger than 1 for $d/R = 2$ and $U/(gR)^{\frac{1}{2}}$ near unity. Our solution is most likely not a good approximation for $U/(gR)^{\frac{1}{2}}$ near unity and d/R smaller than about 4. The figure also indicates that the linearized lee-wave problem may with fair approximation be solved by applying (3.17), instead of the complete integral equation.

We have tacitly assumed that (3.13) has a unique solution, which is confirmed by the numerical results. This is easily shown to be true if we assume that the physical problem is unique, which seems to be obvious when $\tau \neq \frac{1}{4}$. Following an argument given by Ursell (1973) in another context, we assume for the moment that $\partial\phi/\partial n = 0$ (the body at rest) and there is no incoming wave. According to our basic assumption there is then no motion, and $f_1(z)$ is constant outside the body. The right-hand side of (3.13) is zero, and we shall prove that $\gamma = 0$. Let us consider the potential defined

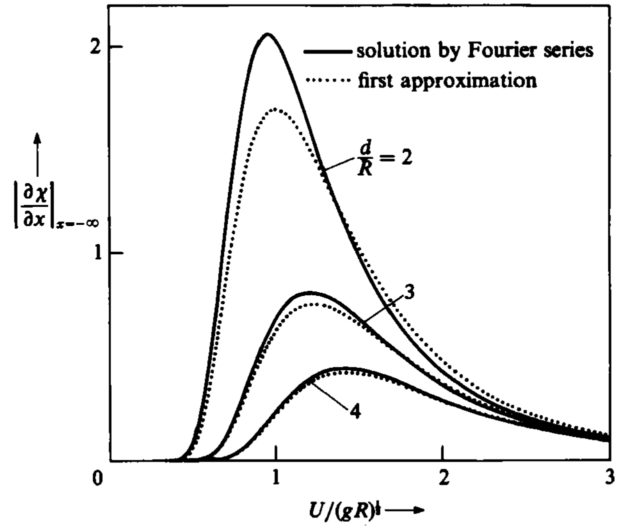

FIGURE 7. Maximum of $|\partial\chi/\partial x|$ at $x=-\infty$, $d/R = 2.0, 3.0, 4.0$.

by (3.8) inside the body. It is immediately seen that $f_1(z)$ is continuous across the boundary. Hence $f_1(z)$ inside the body is constant, i.e. $f_1'(z) = 0$. Approaching the boundary from the inside and applying the Plemelj theorem, we then have

$$\mathrm{Im}_1\,(2f_1'(\zeta(s'))\exp(\mathrm{i}\beta(s'))) = -\gamma(s')+\frac{1}{\pi}\int_S \gamma(s)\,L(s',s)\,\mathrm{d}s = 0. \tag{6.1}$$

Comparing with (3.13) with the right-hand side zero, it follows that $\gamma = 0$. Hence we have proved the assertion.

Appendix A. Solution of the integral equation by Fourier transform

The exact solution of the integral equation (3.13) is obtained using a Fourier transform. The Fourier series (3.20) for the unknown γ and the parameter forms (3.18) and (3.19) for the contour and the angle β are used. Multiplying the integral equation with $(1/2\pi)\exp(-\mathrm{i}n\theta')$ and integrating from 0 to 2π, the following two infinite sets of equations are obtained:

$$\begin{pmatrix} C_n^{(1)}-C_n^{(4)} \\ C_n^{(2)}+C_n^{(3)} \end{pmatrix} + \sum_{m=1}^{\infty} \begin{pmatrix} \alpha_{nm}^{(1)} & -\beta_{nm}^{(1)} \\ \beta_{nm}^{(1)} & \alpha_{nm}^{(1)} \end{pmatrix} \begin{pmatrix} C_m^{(1)}-C_m^{(4)} \\ C_m^{(2)}+C_m^{(3)} \end{pmatrix} = \begin{pmatrix} h_n^{(1)}-h_n^{(4)} \\ h_n^{(2)}+h_n^{(3)} \end{pmatrix}, \tag{A 1}$$

$$\begin{pmatrix} C_n^{(1)}+C_n^{(4)} \\ C_n^{(2)}-C_n^{(3)} \end{pmatrix} + \sum_{m=1}^{\infty} \begin{pmatrix} \alpha_{nm}^{(2)} & -\beta_{nm}^{(2)} \\ \beta_{nm}^{(2)} & \alpha_{nm}^{(2)} \end{pmatrix} \begin{pmatrix} C_m^{(1)}+C_m^{(4)} \\ C_m^{(2)}-C_m^{(3)} \end{pmatrix} = \begin{pmatrix} h_n^{(1)}+h_n^{(4)} \\ h_n^{(2)}-h_n^{(3)} \end{pmatrix}, \tag{A 2}$$

where

$$\begin{aligned}\alpha_{nm}^{(1)}+\mathrm{i}\beta_{nm}^{(1)} &= \frac{(m+n-1)!}{m!(n-1)!}\left(\frac{R}{2d}\right)^{m+n} + \frac{2}{(1-4\tau)^{\frac{1}{2}}}\left(\frac{R}{2d}\right)^{m}\frac{1}{(n-1)!} \\ &\times\left[(Rk_1)^n\left(\exp(-2k_1 d)\,E_{m+1}(2k_1 d\exp(-\mathrm{i}\pi)) + \sum_{q=1}^{n-1}\frac{(m+q-1)!}{m!}\frac{1}{(2k_1 d)^q}\right)\right. \\ &\left. -(Rk_2)^n\left(\exp(-2k_2 d)\,E_{m+1}(2k_2 d\exp(\mathrm{i}\pi)) + \sum_{q=1}^{n-1}\frac{(m+q-1)!}{m!}\frac{1}{(2k_2 d)^q}\right)\right],\end{aligned} \tag{A 3}$$

$$
\begin{aligned}
\alpha^{(2)}_{nm} + \mathrm{i}\beta^{(2)}_{nm}
&= \frac{(m+n-1)!}{m!\,(n-1)!}\left(\frac{R}{2d}\right)^{m+n} + \frac{2}{(1+4\tau)^{\frac{1}{2}}}\left(\frac{R}{2d}\right)^{m}\frac{1}{(n-1)!}\\
&\times\left[(Rk_3)^n\left(\exp(-2k_3 d)\,E_{m+1}(2k_3 d\exp(-\mathrm{i}\pi)) + \sum_{q=1}^{n-1}\frac{(m+q-1)!}{m!}\frac{1}{(2k_3 d)^q}\right)\right.\\
&\left.-(Rk_4)^n\left(\exp(-2k_4 d)\,E_{m+1}(2k_4 d\exp(-\mathrm{i}\pi)) + \sum_{q=1}^{n-1}\frac{(m+q-1)!}{m!}\frac{1}{(2k_4 d)^q}\right)\right],
\end{aligned}
\tag{A 4}
$$

$$C_n = C^{(1)}_n + \mathrm{i}C^{(2)}_n + \mathrm{j}C^{(3)}_n + \mathrm{ij}C^{(4)}_n, \tag{A 5}$$

$$h_n = h^{(1)}_n + \mathrm{i}h^{(2)}_n + \mathrm{j}h^{(3)}_n + \mathrm{ij}h^{(4)}_n. \tag{A 6}$$

h_n is the Fourier transform of the right-hand side of (3.13),

$$h_n = \frac{1}{2\pi}\int_0^{2\pi} h(\theta')\exp(-\mathrm{i}n\theta')\,\mathrm{d}\theta', \tag{A 7}$$

and $C^{(q)}_n$, $h^{(q)}_n$ ($q = 1, 2, 3, 4$) are reals.

The exponential integral E_{m+1} of order $m+1$ is defined in Abramowitz & Stegun (1972) by

$$E_{m+1}(z) = \frac{(-z)^m}{m!}\left(-\ln z - \gamma + \sum_{q=1}^{m}\frac{1}{q}\right) - \sum_{\substack{q=0\\ q\neq m}}^{\infty}\frac{(-z)^q}{q!\,(q-m)}, \tag{A 8}$$

where $\gamma = 0.5772157\ldots$ is Euler's constant.

The systems of equations (A 1) and (A 2) contain the two limits $U\to 0$ ($k_1, k_3 \to \infty$, $k_2 k_4 \to \nu$) and the lee-wave problem $\sigma \to 0$ ($k_1, k_3 \to g/U^2$, $k_2, k_4 \to 0$).

The Fourier transforms of the various right-hand sides are as follows.

The lee-wave problem; $f_0(z) = -Uz$:

$$h_n = \mathrm{i}U\delta_{n1}, \tag{A 9}$$

where δ_{n1} is the Kronecker delta.

The sway problem; $2\dfrac{\partial\phi}{\partial n} = \epsilon\left(-2\mathrm{j}\sigma\sin\theta + 2U\dfrac{\partial^2\chi}{\partial n\,\partial x}\right)$:

$$h_n = \epsilon\left[\mathrm{ij}\sigma\delta_{n1} - \frac{U}{R}n(T_{n-1} - T_{n+1})\right] \quad (n \geqslant 1). \tag{A 10}$$

The heave problem; $2\dfrac{\partial\phi}{\partial n} = \epsilon\left(2\mathrm{j}\sigma\cos\theta + 2U\dfrac{\partial^2\chi}{\partial n\,\partial y}\right)$:

$$h_n = \epsilon\left[\mathrm{j}\sigma\delta_{n1} + \mathrm{i}\frac{U}{R}n(T_{n-1} + T_{n+1})\right] \quad (n \geqslant 1). \tag{A 11}$$

The diffraction problem; $f_0(z) = \delta a\left(\dfrac{g}{k}\right)^{\frac{1}{2}}(1 \pm \mathrm{ij})\exp(-\mathrm{i}kz)$:

$$h_n = -(1 \pm \mathrm{ij})\,\delta a(gk)^{\frac{1}{2}}\exp(-kd)\frac{(kR)^{n-1}}{(n-1)!} \quad (n \geqslant 1). \tag{A 12}$$

In (A 10) and (A 11) the tangential velocity of the lee-wave problem ($T_0 = 0$)

$$v_t = U\sum_{m=1}^{\infty}(T_m\exp(\mathrm{i}m\theta) + \overline{T}_m\exp(-\mathrm{i}m\theta)) \tag{A 13}$$

is used. It may be shown that $UT_n = -\mathrm{i}C_n$, where C_n is the Fourier transform of γ in the lee-wave problem.

The amplitudes of the far-field solution for the velocity potential $f_1(z)$ are given by the following infinite series:

$$A_q = \mathrm{i}(1-\mathrm{ij})\frac{2\pi R\exp(-k_q d)}{(1-4\tau)^{\frac{1}{2}}} \times \sum_{m=1}^{\infty}\frac{(k_q R)^m}{m!}(C_m^{(1)}-C_m^{(4)}+\mathrm{i}(C_m^{(2)}+C_m^{(3)})) \quad (q=1,2), \quad \text{(A 14)}$$

$$A_q = \mathrm{i}(1+\mathrm{ij})\frac{2\pi R\exp(-k_q d)}{(1+4\tau)^{\frac{1}{2}}} \times \sum_{m=1}^{\infty}\frac{(k_q R)^m}{m!}(C_m^{(1)}+C_m^{(4)}+\mathrm{i}(C_m^{(2)}-C_m^{(3)})) \quad (q=3,4). \quad \text{(A 15)}$$

Appendix B. The energy flux

Let us consider a single harmonic wavetrain. Retaining only terms to the second order in the amplitude, we have

$$R_x = \overline{\int_{-h}^{\eta}(p+\tfrac{1}{2}\rho v^2+\rho g y)(-U+u^{(1)}+u^{(2)})\,\mathrm{d}y}, \quad \text{(B 1)}$$

where h is the depth of the fluid layer and a bar denotes time average. The other quantities are defined in §4. We may write

$$p = -\rho(\hat{\Phi}_t+\tfrac{1}{2}(v^2-U^2)+gy)+\hat{p}, \quad \text{(B 2)}$$

where $\hat{\Phi}$ is the velocity potential to second order with $\hat{\Phi}_t = 0$, and $\hat{p}$ is a second-order constant. Hereby

$$R_x = \overline{\int_{-h}^{\eta}(-\rho\hat{\Phi}_t+\tfrac{1}{2}\rho U^2+\hat{p})(-U+u^{(1)}+u^{(2)})\,\mathrm{d}y} = \overline{\int_{-h}^{\eta}-\rho\hat{\Phi}_t(-U+u^{(1)})\,\mathrm{d}y}+\left(\tfrac{1}{2}U^2+\frac{\hat{p}}{\rho}\right)M, \quad \text{(B 3)}$$

where M is the mean mass flux:

$$M = \rho\overline{\int_{-h}^{\eta}(-U+u^{(1)}+u^{(2)})\,\mathrm{d}y}. \quad \text{(B 4)}$$

Now
$$\overline{\int_{-h}^{\eta}-\rho\hat{\Phi}_t(-U+u^{(1)})\,\mathrm{d}y} = \rho\overline{\hat{\Phi}_t\eta^{(1)}}U-\rho\int_{-h}^{0}\overline{\hat{\Phi}_t u^{(1)}}\,\mathrm{d}y. \quad \text{(B 5)}$$

After some elementary algebra we find

$$R_x = \left(\tfrac{1}{2}U^2+\frac{\hat{p}}{\rho}\right)M+E\frac{c-U}{c}(c_g-U), \quad \text{(B 6)}$$

where c and c_g are respectively the phase velocity and group velocity in the relative frame of reference.

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